

A New Approach for Helper Selection and Performance Analysis in Poisson CoopMAC Networks

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Abstract

The cooperative medium access control (CoopMAC) protocol in the presence of randomly-distributed nodes and shadowing is considered. The nodes are assumed to be distributed according to a homogeneous two-dimensional Poisson point process. A new approach is proposed for helper selection and throughput performance analysis which depends on the shadowing parameters as well as the distribution of helpers. In the proposed protocol, the potential helpers are divided into several tiers based on their distances from the source and destination in a way that the lower the tier index, the higher its priority. When there are several helpers of the same tier, the helper that is less affected by shadowing is chosen for cooperation. Upper and lower bounds are derived for the average cooperative throughput of the proposed CoopMAC protocol. It is observed that the proposed scheme readily outperforms the conventional CoopMAC protocol in having larger average throughput. It is also seen that the cooperative throughput of the proposed scheme approaches the upper bound when the density of nodes increases.

Index Terms

Cooperative medium access control (CoopMAC), helper selection, IEEE 802.11b, Poisson point process, shadowing, stochastic geometry.

I. INTRODUCTION

The broadcast nature of the wireless medium is one of the most important features of wireless communication networks. A direct consequence of this feature is that the transmission between

any two nodes can be overheard by other nodes of the network. As a result, a source node can make use of the other nodes (referred to as *helpers* in the sequel) to improve performance measures such as throughput, bit error rate (BER) and diversity gain.

In recent years, many studies have been conducted on cooperation between nodes in the medium access control (MAC) layer. Some of the protocols proposed for the MAC layer are based on the IEEE 802.11b Standard [1]. In this standard, a multirate scheme is employed for establishing a connection between two nodes in a wireless local area network (WLAN). An important drawback of the above multirate scheme is the fact that for transmitting the same amount of information, low-rate links make the channel busy for a longer time than the high-rate links. A possible approach to overcome this deficiency is to make use of an appropriate helper to retransmit the source signal to destination and reduce the total transmission time, i.e., a cooperative MAC (CoopMAC) protocol is used [2].

In a CoopMAC protocol, the nodes are assumed to be distributed in a given region and each node keeps a table (known as CoopTable) containing the information corresponding to the helpers that can possibly assist the source during its transmission [2]. Before transmitting its packet, each node looks up the CoopTable to see if there is a helper that can improve the overall transmission rate. If there are several such helpers, the one with the latest feedback time is chosen for cooperation. The feedback time is the latest time a successful transmission is observed from that helper. In [3], a similar protocol to CoopMAC has been proposed in which two helpers with the latest feedback times are chosen to assist the source. Interestingly, it is shown in reference [4] that a high-rate node assisting a low-rate node can improve its own throughput, delay and energy consumption. This is because by forwarding a low-rate node's data, the high-rate node can gain access to a free channel in a shorter time to complete its own transmission [4]. Observe that the creation and maintenance of the CoopTable needs additional memory at each node and increases the complexity of the system significantly [5]. Moreover, the performance of CoopMAC protocols which are based on the CoopTable severely degrades when the helpers are mobile. In order to address this problem, a new protocol based on CoopMAC has been proposed in [6] which separates the mobile helpers from the static ones by maintaining a history of the signal strength corresponding to each helper's overheard packets.

In [7], a new cooperation protocol known as persistent relay carrier sensing multiple access (PRCSMA) has been proposed which employs an automatic retransmission request (ARQ)

scheme to enhance the overall performance of the IEEE 802.11 protocol. In this protocol, each time a packet is received with error, the destination automatically transmits a claim for cooperation (CFC) packet to the other nodes, and requests for a retransmission of the original packet. In this protocol, all idle nodes can act as a potential helper as long as they satisfy a set of relay selection conditions. PRCSMA protocol is known to improve the channel usage and to increase the transmission range [7], however, it has poor bandwidth efficiency [5].

In [8], a variation of the CoopMAC protocol has been proposed in which cooperation takes place only when there exists a potential cooperative link which can provide a desired transmission rate that cannot be achieved by the direct link. This protocol, however, suffers from high computational complexity as the link capacity is estimated using instantaneous signal-to-noise power ratio (SNR), and the latter requires two channel state estimations (namely, source to helper and helper to destination channels) for each potential helper [9]. Note that CoopMAC has been also investigated from other viewpoints in the literature. In [10], it is shown that the network lifetime can be improved by employing CoopMAC. Also in [11], a game theoretical approach has been proposed for analyzing a CoopMAC protocol with incentive design.

An important assumption which seems to be less investigated in the literature is the effect of shadowing on the performance of the CoopMAC-based protocols [12]. Shadowing occurs when there are obstacles that block the line-of-sight (LOS) path between two communicating nodes and can attenuate the transmitted signal power drastically. Therefore, it is a major impairment in wireless networks and must be taken into account in the design and evaluation of these networks [12]–[14]. In [12], a new CoopMAC protocol has been proposed and the effect of uncorrelated shadowing on the average number of nodes that can receive a packet with desired quality of service (QoS) has been examined. In addition, the effect of correlated shadowing on the number of helpers that are capable of cooperation in a two-way network-coded (NC) relaying system has been studied in [14].

Motivated by the above facts, in this paper we propose a new helper selection scheme for a CoopMAC network whose nodes are distributed according to a homogeneous two-dimensional Poisson point process (PPP) with a fixed density [15], [16]. To the authors knowledge, the effect of random spatial distribution of the nodes on the overall throughput of the CoopMAC networks seems to have received little attention in the literature. We assume the communication between any two nodes is subject to path loss and shadowing and derive exact expression

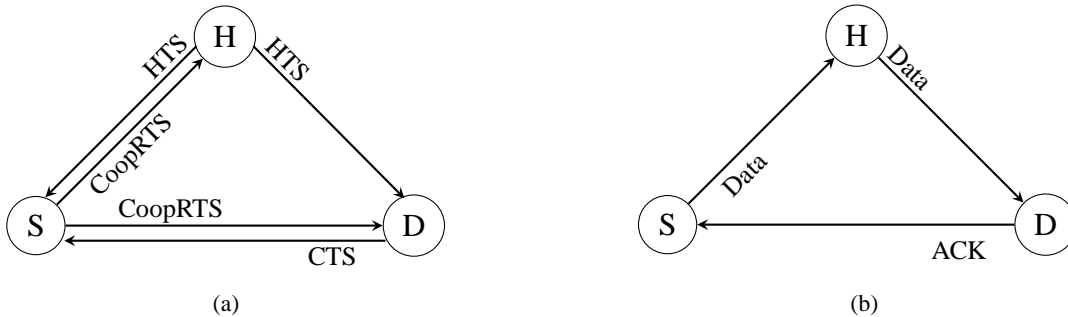


Fig. 1. Illustration of (a) control frame exchange, and (b) data frame exchange, in a typical CoopMAC link.

for the throughput of the direct link between two arbitrary nodes in a random CoopMAC network.¹ We also find upper and lower bounds for the throughput of a cooperative link making use of our proposed helper selection scheme in the presence of shadowing. Our numerical results demonstrate that the proposed CoopMAC has superior throughput performance, and its throughput is only slightly smaller than the upper bound in all the examined scenarios.

The rest of this paper is organized as follows. In Section II, the system model is introduced. We classify the links based on their corresponding source-destination distance and propose our helper selection scheme and data transmission framework in Section III. In Sections IV, V and VI, a theoretical throughput analysis is presented for each of the link types introduced in Section III. Numerical results are provided in Section VII to verify the superiority of our proposed scheme over the conventional CoopMAC protocol. Concluding remarks are presented in Section VIII.

II. SYSTEM MODEL

In this section, we first provide a brief description on how a CoopMAC protocol works. Then, the effect of shadowing on the probability of a successful transmission is examined.

A. The CoopMAC Protocol

We consider a wireless network whose nodes constitute a homogeneous two dimensional PPP with density λ . The nodes communicate with each other according to a carrier sense multiple access with collision avoidance (CSMA/CA) scheme in the IEEE 802.11b Standard in distributed

¹We assume in the following that in a *Poisson CoopMAC network*, the locations of the nodes constitute a two-dimensional PPP with a fixed density.

TABLE I
THE LINK TYPES AND THEIR CORRESPONDING TRANSMISSION RATES IN THE IEEE 802.11B STANDARD ($\text{BER} \geq 10^{-5}$).

Link Type	d_{SD} (meter)	Transmission Rate
\mathcal{A}	$0 \leq d_{SD} < 48.2$	11 Mbps
\mathcal{B}	$48.2 \leq d_{SD} < 67.1$	5.5 Mbps
\mathcal{C}	$67.1 \leq d_{SD} < 74.7$	2 Mbps
\mathcal{D}	$74.7 \leq d_{SD} \leq 100$	1 Mbps

coordination function (DCF) mode [1]. We assume the propagation delay is small enough to ensure that CSMA/CA provides a collision-free environment for destination. Our network makes use of the RTS/CTS technique, in which each node can distinguish whether the received packet is for itself or should be forwarded to another node.

A typical CoopMAC link making use of an RTS/CTS scheme is shown in Fig. 1 [2]. Before any communication, the best potential helper must be selected based on a set of criteria. Then, the source node transmits a cooperative RTS (CoopRTS) packet to the best helper and reserves a channel. If the helper is willing to cooperate, it sends back a helper-ready-to-send (HTS) packet. Moreover, the destination node confirms the reservation of a channel by transmitting a CTS packet to the source [2]. If the source node receives both HTS and CTS packets, the cooperation starts and the helper forwards the data packets to the destination through cooperative link. If only the CTS is received, the transmission is made through the direct link. If neither CTS nor HTS packets are received during a specific time, a timeout occurs and the transmission is declared as failed. The transmission is considered as successful when the source node receives an acknowledgment (ACK) packet from destination.

We assume in the sequel that d_{SH} , d_{HD} and d_{SD} represent the source-to-helper, helper-to-destination and source-to-destination distances, respectively.² Moreover, R_{SH} , R_{HD} and R_{SD} denote the transmission rates of source-helper (S–H), helper-destination (H–D) and source-destination (S–D) links, respectively.

²Throughout this paper, all distances are expressed in meters.

B. Effect of Shadowing on Successful Transmission Probability

Shadowing is referred to case where the received signal power is affected by the objects obstructing the path between transmitter and receiver. In order to model the path loss plus shadowing effects, we assume that the received power in dB at destination node is given by [17, eq. 2.51]

$$P_r = P_t + K - 10 \alpha \log_{10}(d_{SD}) + \psi, \quad (1)$$

where P_t is the transmitted power in dB and is assumed to be the same for all nodes (including the helpers), K is a constant in dB which depends on the antenna characteristics, α is the path loss exponent usually between 2 and 7, and ψ is a Gaussian random variable in dB units which represents the effect of shadowing and has mean zero and standard deviation σ_ψ . Depending on the quality of service (QoS) requirements, a threshold for the received power in dB, i.e., P_{th} , is defined at the destination. Therefore, the probability of a successful transmission through the direct link between two nodes at distance d_{SD} equals

$$\mathcal{P}_{\text{Direct}}^{\text{Succ}} = \Pr\{P_r \geq P_{th}\}. \quad (2)$$

Substituting for P_r from (1) into (2), we obtain

$$\begin{aligned} \mathcal{P}_{\text{Direct}}^{\text{Succ}} &= \Pr\{\psi \geq P_{th} - P_t - K + 10 \alpha \log_{10}(d_{SD})\} \\ &= \mathbb{Q}(\nu + \mu \log_{10}(d_{SD})) \end{aligned} \quad (3a)$$

where

$$\nu = \frac{P_{th} - P_t - K}{\sigma_\psi} \quad (3b)$$

$$\mu = \frac{10\alpha}{\sigma_\psi} \quad (3c)$$

and $\mathbb{Q}(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2} du$ is the Gaussian \mathbb{Q} -function. Note that in our treatment, the maximum effective transmission range equals 100 meters. Therefore, it is reasonable to assume that the argument of \mathbb{Q} -function in (3a) is always negative, or analogously $\mathcal{P}_{\text{Direct}}^{\text{Succ}} > 0.5$ for $0 < d_{SD} \leq 100$.

Similarly, for a cooperative link (i.e., a link whose source and destination nodes communicate

TABLE II
THE HELPER TIERS AND TRANSMISSION RATES FOR A TYPE C LINK.

Tier	d_{SH} (meter)	R_{SH} (Mbps)	d_{HD} (meter)	R_{HD} (Mbps)	R_{Coop} (Mbps)
1	$[0, 48.2)$	11	$[0, 48.2)$	11	5.5
2	$[0, 48.2)$	11	$[48.2, 67.1)$	5.5	3.67
	$[48.2, 67.1)$	5.5	$[0, 48.2)$	11	
3	$[48.2, 67.1)$	5.5	$[48.2, 67.1)$	5.5	2.75

through a helper) the probability of a successful transmission can be obtained as

$$\mathcal{P}_{Coop}^{Succ} = \Pr\{P_{r,H} \geq P_{th}\} \Pr\{P_{r,D} \geq P_{th}\} = \mathbb{G}(d_{SH}, d_{HD}) \quad (4a)$$

where

$$\mathbb{G}(\ell_1, \ell_2) = \mathbb{Q}(\nu + \mu \log_{10}(\ell_1)) \mathbb{Q}(\nu + \mu \log_{10}(\ell_2)) \quad (4b)$$

and $P_{r,H}$ and $P_{r,D}$ are the received powers at the helper and destination, respectively.

III. NETWORK CLASSIFICATION AND BEST HELPER SELECTION ALGORITHM

In the IEEE 802.11b Standard, the closer the source and destination nodes the higher the transmission rate [1]. As a result, in a CoopMAC network based on the IEEE 802.11b Standard, we can define four types of links depending on the distance between the source and destination nodes as illustrated in Table I. In what follows we investigate whether a helper can increase the transmission rate corresponding to each of the link types listed in Table I or not.

1) *Type A Links*: The transmission rate for this type is 11 Mbps which is the maximum transmission rate in the IEEE 802.11b Standard. In consequence, a helper is not used for the links of this type.

2) *Type B Links*: Similar to Type A links, for Type B links a helper cannot improve the transmission rate. To show this, we assume that the source wants to transmit L bits of information to destination through a helper. Denoting the transmission times of S-H and H-D links by t_{SH} and t_{HD} , respectively, we can obtain the cooperation time as

$$t_{Coop} = t_{SH} + t_{HD}. \quad (5)$$

TABLE III
THE HELPER TIERS AND TRANSMISSION RATES FOR A TYPE \mathcal{D} LINK.

Tier	d_{SH} (meter)	R_{SH} (Mbps)	d_{HD} (meter)	R_{HD} (Mbps)	R_{Coop} (Mbps)
1	[0, 48.2)	11	[0, 48.2)	11	5.5
2	[0, 48.2)	11	[48.2, 67.1)	5.5	3.67
	[48.2, 67.1)	5.5	[0, 48.2)	11	
3	[48.2, 67.1)	5.5	[48.2, 67.1)	5.5	2.75
4	[0, 48.2)	11	[67.1, 74.7)	2	1.69
	[67.1, 74.7)	2	[0, 48.2)	11	
5	[48.2, 67.1)	5.5	[67.1, 74.7)	2	1.47
	[67.1, 74.7)	2	[48.2, 67.1)	5.5	

It is clear that the time taken for transmission of a sequence of L bits³ over a link with a transmission rate of R bps equals L/R . Thus, the overall rate of the cooperative link (i.e., S–H–D link), is given by [6, eq. (1)]

$$R_{Coop} = \frac{L}{t_{Coop}} = \frac{L}{\frac{L}{R_{SH}} + \frac{L}{R_{HD}}} = \frac{R_{SH} R_{HD}}{R_{SH} + R_{HD}} \quad (6)$$

where R_{SH} and R_{HD} are the S–H and H–D link rates. In order to obtain the maximum value of R_{Coop} , one should minimize the denominator of the fraction on the right of (6). Clearly, the minimum is attained when $R_{SH} = R_{HD} = 11$ Mbps and, thus, the maximum rate of the cooperative link becomes 5.5 Mbps which equals the transmission rate of the direct link. Hence, even in the best-case scenario a helper cannot improve the transmission rate of a Type \mathcal{B} link.

3) *Type C Links*: For the links of this type, there are four different cases where cooperation is beneficial, i.e., the overall transmission rate is greater than 2 Mbps. These cases lead to three different helper tiers, viz.,

Tier 1: Both d_{SH} and d_{HD} are less than 48.2, and R_{Coop} is equal to 5.5 Mbps.

Tier 2: Either d_{SH} or d_{HD} is less than 48.2 and the other is in [48.2, 67.1) range. R_{Coop} equals 3.67 Mbps.

Tier 3: Both d_{SH} and d_{HD} are in [48.2, 67.1) range, and $R_{Coop} = 2.75$ Mbps.

³This sequence is assumed to include the RTS and CTS packets as well.

The above definitions are summarized in Table II. Clearly, for Type \mathcal{C} links, a helper is useful only when neither R_{SH} nor R_{HD} are smaller than 5.5 Mbps. Note that the larger the tier index the smaller the cooperative rate (R_{Coop}).

4) *Type \mathcal{D} Links*: There are eight cases where a helper can improve the overall transmission rate of a Type \mathcal{D} link. As shown in Table III, these cases result in five different tiers of helpers. Observe that the specifications of Tiers 1 through 3 helpers are the same for both Type \mathcal{C} and Type \mathcal{D} links. The specifications of Tier 4 and Tier 5 helpers are given in Table III. Again, as the tier index increases, the corresponding cooperative rate decreases. Note, importantly, that a Type \mathcal{D} link is established between two nodes whose distance is between 74.7 and 100 meters. Thus, a Tier 1 helper does not exist when the source and destination nodes are more than 96.4 meters apart.

In summary, a helper in a CoopMAC network is beneficial when the following two conditions are satisfied:

- 1) The link between the source and destination nodes is of Type \mathcal{C} or \mathcal{D} .
- 2) The transmission rate of the cooperative link (R_{Coop}) exceeds that of the direct link.

Suppose now that the S–D link is of Type \mathcal{C} . Then, a list of Tier 1 helpers denoted by \mathcal{H}_1 , is created. If $\mathcal{H}_1 \neq \emptyset$, the helper with the largest $\mathcal{P}_{Coop}^{Succ}$ in \mathcal{H}_1 , is selected for cooperation. The CoopRTS/CTS handshake is then accomplished as explained in Subsection II-A. When \mathcal{H}_1 is empty, a list of Tier 2 helpers (i.e., \mathcal{H}_2) is formed and, again, the helper with the largest $\mathcal{P}_{Coop}^{Succ}$ in \mathcal{H}_2 , is selected for cooperation. Similarly, when both \mathcal{H}_1 and \mathcal{H}_2 are empty, a list of Tier 3 helpers is created, and cooperation is done through a helper in \mathcal{H}_3 that has the largest $\mathcal{P}_{Coop}^{Succ}$. A detailed explanation of the helper selection and transmission procedures for a Type \mathcal{C} link is given in Algorithm 1. Note that Algorithm 1 can be readily modified to be used for the case where the S–D link is of Type \mathcal{D} .

IV. PERFORMANCE ANALYSIS IN THE PRESENCE OF SHADOWING

In this section, we evaluate the throughput of the link types shown in Table I in the presence of shadowing and path loss. For a link whose transmission rate equals R bps the throughput is given by

$$\mathcal{T} = \overline{\mathcal{P}}^{Succ} \cdot R \quad (7)$$

Algorithm 1 Best helper selection and data transmission for a Type \mathcal{C} link

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1: Initialization:
2: if there is any Tier 1 helper then
3:   create  $\mathcal{H}_1$ , i.e., a list of Tier 1 helpers
4: end if
5: select the helper from  $\mathcal{H}_1$  with the largest  $\mathbb{G}(d_{\text{SH}}, d_{\text{HD}})$  and remove it from  $\mathcal{H}_1$ 
6: send a CoopRTS packet to the helper chosen in Step 5
7: if an HTS packet is received then goto Step 27
8: else if  $\mathcal{H}_1$  is not empty then goto Step 5
9: end if
10: if there is any Tier 2 helper then
11:   create  $\mathcal{H}_2$ , i.e., a list of Tier 2 helpers
12: end if
13: select the helper from  $\mathcal{H}_2$  with the largest  $\mathbb{G}(d_{\text{SH}}, d_{\text{HD}})$  and remove it from  $\mathcal{H}_2$ 
14: send a CoopRTS packet to the helper chosen in Step 13
15: if an HTS packet is received then goto Step 27
16: else if  $\mathcal{H}_2$  is not empty then goto Step 13
17: end if
18: if there is any Tier 3 helper then
19:   create  $\mathcal{H}_3$ , i.e., a list of Tier 3 helpers
20: end if
21: select the helper from  $\mathcal{H}_3$  with the largest  $\mathbb{G}(d_{\text{SH}}, d_{\text{HD}})$  and remove it from  $\mathcal{H}_3$ 
22: send a CoopRTS packet to the helper chosen in Step 21
23: if an HTS packet is received then goto Step 27
24: else if  $\mathcal{H}_3$  is not empty then goto Step 21
25: end if
26: send an RTS packet
27: if a CTS packet is not received then
28:   perform a random backoff and goto Step 1
29: end if
30: send data
31: if an ACK packet is not received then
32:   perform a random backoff and goto Step 1
33: end if
34: Transmission Complete

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where $\overline{\mathcal{P}}^{\text{Succ}}$ is the average probability of a successful transmission through this link. We use eq. (7) in the following to find the throughputs of the link types presented in Table I.

A. Type \mathcal{A} and Type \mathcal{B} Links

As mentioned earlier, for Type \mathcal{A} and Type \mathcal{B} links a helper cannot improve the overall transmission rate. Therefore, for these link types a helper is not used. Assume that the source node is the k th nearest neighbor of the destination node and $d_{SD} = r_k$. When the nodes are distributed according to a two dimensional homogeneous PPP with density λ , the probability density function (PDF) of r_k is given by [18]

$$f_{r_k}(r) = 2e^{-\lambda\pi r^2} \frac{(\lambda\pi r^2)^k}{r(k-1)!} u(r) \quad (8)$$

where $u(r)$ is the unit step function. Thus, using (3a) along with (8), we can find the average probability of a successful transmission for Type \mathcal{A} and Type \mathcal{B} links as $\overline{\mathcal{P}}_{\mathcal{A}}^{\text{Succ}} = \mathbb{H}(0, 48.2)$ and $\overline{\mathcal{P}}_{\mathcal{B}}^{\text{Succ}} = \mathbb{H}(48.2, 67.1)$, respectively, where

$$\mathbb{H}(r_{\min}, r_{\max}) \triangleq \int_{r_{\min}}^{r_{\max}} \mathbb{Q}(\nu + \mu \log_{10}(r)) f_{r_k}(r) dr. \quad (9)$$

Hence, using Table I and eq. (7) we can obtain the throughputs of Type \mathcal{A} and Type \mathcal{B} links as

$$\mathcal{T}^{\mathcal{A}} = \overline{\mathcal{P}}_{\mathcal{A}}^{\text{Succ}} \times 11 \text{ (Mbps)} \quad (10)$$

$$\mathcal{T}^{\mathcal{B}} = \overline{\mathcal{P}}_{\mathcal{B}}^{\text{Succ}} \times 5.5 \text{ (Mbps)} \quad (11)$$

respectively.

B. Type \mathcal{C} and Type \mathcal{D} Links

For Type \mathcal{C} and \mathcal{D} links a helper may or may not be used as explained in Section II. Assume now that there is no helper for cooperation and the transmission is made through the direct link. Then, similar to Type \mathcal{A} and \mathcal{B} links, the throughput of Type \mathcal{C} and \mathcal{D} links can be obtained, respectively, as

$$\mathcal{T}_{\text{Direct}}^{\mathcal{C}} = \mathbb{H}(67.1, 74.7) \times 2 \text{ (Mbps)} \quad (12)$$

$$\mathcal{T}_{\text{Direct}}^{\mathcal{D}} = \mathbb{H}(74.7, 100) \times 1 \text{ (Mbps)}. \quad (13)$$

When a helper is available for cooperation, the resulting throughput (referred to as cooperative throughput) equals $R_{\text{Coop}} \mathcal{P}_{\text{Coop}}^{\text{Succ}}$ where R_{Coop} is given in Table II for a Type \mathcal{C} link and in Table III

for a Type \mathcal{D} link, and $\mathcal{P}_{\text{Coop}}^{\text{Succ}}$ was defined in (4a). Observe that the cooperative throughput depends on the link type as well as the tier of the helper. Since the helpers are randomly distributed in the S–D plane, the average cooperative throughput is obtained by averaging $R_{\text{Coop}} \mathcal{P}_{\text{Coop}}^{\text{Succ}}$ over the spatial distribution of the helper nodes. This can become quite complicated as it requires the joint PDF of d_{SH} and d_{HD} which is not easy to obtain. Moreover, the final result involves a three-fold integration which is difficult to evaluate. Therefore, in the next two sections we derive upper and lower bounds on $\mathcal{P}_{\text{Coop}}^{\text{Succ}}$ for Type \mathcal{C} and \mathcal{D} links and use these bounds to subsequently derive upper and lower bounds on the average cooperative throughputs of these links.

Before proceeding further, we establish a fact which will be used in the sequel to obtain the probability that a helper of a given tier can be found for a Type \mathcal{C} or \mathcal{D} link. Assume that we have a field of nodes distributed in a region \mathcal{R} according to a two-dimensional PPP with density λ . Also assume that \mathcal{A} is a subregion of \mathcal{R} , i.e., $\mathcal{A} \subseteq \mathcal{R}$. Then the probability that a node \mathbf{X} is located in \mathcal{A} , provided that it is located in \mathcal{R} equals [19, Def. 3.2–(ii)]

$$\Pr\{\mathbf{X} \in \mathcal{A} | \mathbf{X} \in \mathcal{R}\} = \frac{\mathcal{S}(\mathcal{A})}{\mathcal{S}(\mathcal{R})} \quad (14)$$

where $\mathcal{S}(\mathcal{A})$ and $\mathcal{S}(\mathcal{R})$ are the surface areas of \mathcal{A} and \mathcal{R} , respectively.

V. BOUNDS ON THE AVERAGE THROUGHPUT OF A TYPE \mathcal{C} LINK

In this section we evaluate the maximum and minimum cooperative throughputs of a Type \mathcal{C} link. In the following subsections, we assume $\mathcal{P}_{\text{Coop}}^{\text{Succ}, \mathcal{C}, i}$ and $\mathcal{T}_{\text{Coop}}^{\mathcal{C}, i}$ to be, respectively, the probability of a successful transmission and the throughput of a Type \mathcal{C} link making use of a Tier i helper where $i = 1, 2$ and 3 as given in Table II.

A. The Cooperative Throughput Using a Tier 1 Helper

Fig. 2 illustrates a Type \mathcal{C} link in which the source and destination nodes communicate through a Tier 1 helper as shown in Table II. Clearly, the helper has to be located in the shaded area, \mathcal{U}_1 , i.e., the intersection of two circles centered at S and D both with radius 48.2. Since S is the k th nearest neighbor of D, there should be exactly $k - 1$ nodes in a circle centered at D with radius r_k (this region is referred to as \mathcal{W} in Fig. 2). Denoting by $\mathbb{N}(\mathcal{U}_1)$ the number of nodes in \mathcal{U}_1 and using the fact that $\mathcal{U}_1 \subseteq \mathcal{W}$ along with (14), one can find the probability that at least

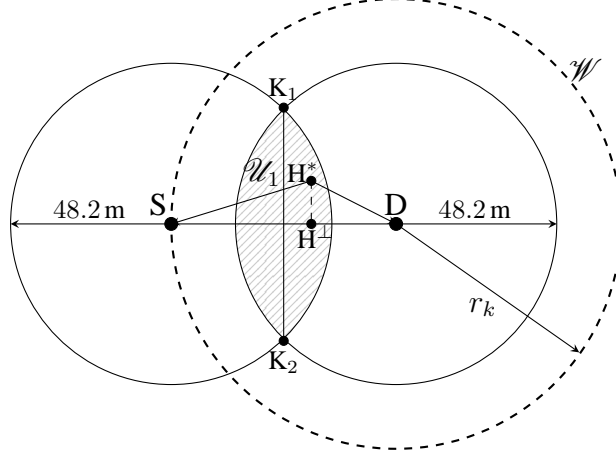


Fig. 2. A typical Type \mathcal{C} link making use of a Tier 1 helper, $67.1 \leq r_k \leq 74.7$.

one Tier 1 helper is available for a Type \mathcal{C} link as

$$\begin{aligned}
 \mathcal{P}_{\mathcal{C},1} &= \Pr\{\mathbb{N}(\mathcal{U}_1) \geq 1\} = 1 - \Pr\{\mathbb{N}(\mathcal{U}_1) = 0\} \\
 &= 1 - \Pr\{\mathbb{N}(\mathcal{W} - \mathcal{U}_1) = k - 1\} \\
 &= 1 - \left(\frac{\mathcal{S}(\mathcal{W}) - \mathcal{S}(\mathcal{U}_1)}{\mathcal{S}(\mathcal{W})} \right)^{k-1} = 1 - \left(1 - \frac{\mathcal{S}(\mathcal{U}_1)}{\pi r_k^2} \right)^{k-1} \quad (15)
 \end{aligned}$$

where $\mathcal{S}(\mathcal{U}_1) = \mathbb{A}(48.2, 48.2, r_k)$ and $\mathbb{A}(r_1, r_2, \ell)$ is the surface area of the intersection of two circles with radii r_1 and r_2 whose centers are ℓ meters apart, that is [2]

$$\mathbb{A}(r_1, r_2, \ell) = r_1^2 \arcsin\left(\frac{h}{r_1}\right) + r_2^2 \arcsin\left(\frac{h}{r_2}\right) - h\ell, \quad (16a)$$

where

$$h = \frac{\sqrt{2r_1^2 r_2^2 + 2(r_1^2 + r_2^2)\ell^2 - (r_1^4 + r_2^4)\ell^4}}{2\ell}. \quad (16b)$$

We now state and prove a lemma which gives the bounds on the cooperative throughput that can be achieved in this case.

Lemma 1: The cooperative throughput of a Type \mathcal{C} link making use of a Tier 1 helper can be bounded as

$$\mathcal{L}_{\text{Coop}}^{\mathcal{C},1} \leq \mathcal{T}_{\text{Coop}}^{\mathcal{C},1} \leq \mathcal{U}_{\text{Coop}}^{\mathcal{C},1} \quad (17a)$$

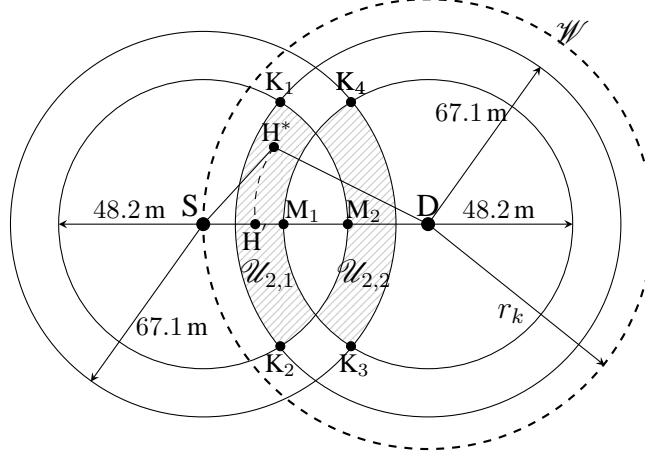


Fig. 3. A typical Type C link making use of a Tier 2 helper, $67.1 \leq r_k \leq 74.7$.

where

$$\mathcal{L}_{\text{Coop}}^{\mathcal{C},1} \triangleq \mathbb{G}(48.2, 48.2) \times 5.5 \text{ (Mbps)} \quad (17b)$$

$$\mathcal{U}_{\text{Coop}}^{\mathcal{C},1} \triangleq \mathbb{G}\left(\frac{r_k}{2}, \frac{r_k}{2}\right) \times 5.5 \text{ (Mbps)}. \quad (17c)$$

Proof: The proof of Lemma 1 is given in Appendix A. ■

B. The Cooperative Throughput Using a Tier 2 Helper

Fig. 3 shows a typical Type C link making use of a Tier 2 helper as shown in Table II. The probability that cooperation is made through a Tier 2 helper, referred to as $\mathcal{P}_{\mathcal{C},2}$, equals the probability that there is no helper in \mathcal{U}_1 , and at least one helper is located in $\mathcal{U}_2 = \mathcal{U}_{2,1} \cup \mathcal{U}_{2,2}$. Using the fact that there are exactly $k - 1$ nodes in a circle centered at D with radius r_k (\mathcal{W} in Fig. 3) one can obtain

$$\begin{aligned} \mathcal{P}_{\mathcal{C},2} &= \Pr\{\mathbb{N}(\mathcal{U}_2) \geq 1 \text{ and } \mathbb{N}(\mathcal{U}_1) = 0\} \\ &= \Pr\{\mathbb{N}(\mathcal{U}_1) = 0\} - \Pr\{\mathbb{N}(\mathcal{U}_1) = 0 \text{ and } \mathbb{N}(\mathcal{U}_2) = 0\} \\ &= \Pr\{\mathbb{N}(\mathcal{W} - \mathcal{U}_1) = k - 1\} - \Pr\{\mathbb{N}(\mathcal{W} - (\mathcal{U}_1 \cup \mathcal{U}_2)) = k - 1\} \\ &= \left(\frac{\mathcal{S}(\mathcal{W}) - \mathcal{S}(\mathcal{U}_1)}{\mathcal{S}(\mathcal{W})}\right)^{k-1} - \left(\frac{\mathcal{S}(\mathcal{W}) - (\mathcal{S}(\mathcal{U}_1) + \mathcal{S}(\mathcal{U}_2))}{\mathcal{S}(\mathcal{W})}\right)^{k-1} \\ &= \left(1 - \frac{\mathcal{S}(\mathcal{U}_1)}{\pi r_k^2}\right)^{k-1} - \left(1 - \frac{\mathcal{S}(\mathcal{U}_1) + \mathcal{S}(\mathcal{U}_2)}{\pi r_k^2}\right)^{k-1} \end{aligned} \quad (18)$$

where $\mathcal{S}(\mathcal{U}_2) = 2(\mathbb{A}(67.1, 48.2, r_k) - \mathbb{A}(48.2, 48.2, r_k))$, and $\mathbb{A}(r_1, r_2, \ell)$ was defined in (16a) and (16b).

We now use a procedure similar to that presented in Lemma 1 to obtain the maximum and minimum of $\mathcal{P}_{\text{Coop}}^{\text{Succ}, \mathcal{C}, 2}$. This procedure is summarized in the following lemma.

Lemma 2: For a Type \mathcal{C} link making use of a Tier 2 helper the cooperative throughput is bounded as

$$\mathcal{L}_{\text{Coop}}^{\mathcal{C}, 2} \leq \mathcal{T}_{\text{Coop}}^{\mathcal{C}, 2} \leq \mathcal{U}_{\text{Coop}}^{\mathcal{C}, 2} \quad (19a)$$

where

$$\mathcal{L}_{\text{Coop}}^{\mathcal{C}, 2} \triangleq \mathbb{G}(48.2, 67.1) \times 3.67 \text{ (Mbps)} \quad (19b)$$

$$\mathcal{U}_{\text{Coop}}^{\mathcal{C}, 2} \triangleq \mathbb{G}(48.2, r_k - 48.2) \times 3.67 \text{ (Mbps)}. \quad (19c)$$

Proof: The proof of Lemma 2 is given in Appendix B. ■

C. The Cooperative Throughput Using a Tier 3 Helper

A typical Type \mathcal{C} link making use of a Tier 3 helper is illustrated in Fig. 4. A Tier 3 helper can be located either in $\mathcal{U}_{3,1}$ or in $\mathcal{U}_{3,2}$ (the shaded areas in Fig 4). These areas are characterized as

$$48.2 \leq d_{\text{SH}} \leq 67.1 \quad (20a)$$

$$48.2 \leq d_{\text{HD}} \leq 67.1 \quad (20b)$$

$$r_k \leq d_{\text{SH}} + d_{\text{HD}}. \quad (20c)$$

A Tier 3 helper is used for cooperation when there is no helper in \mathcal{U}_1 and \mathcal{U}_2 , and there exists at least one helper in $\mathcal{U}_3 \triangleq \mathcal{U}_{3,1} \cup \mathcal{U}_{3,2}$. The probability of this event, referred to as $\mathcal{P}_{\mathcal{C},3}$, can be obtained

$$\begin{aligned} \mathcal{P}_{\mathcal{C},3} &= \Pr\{\mathbb{N}(\mathcal{U}_3) \geq 1 \text{ and } \mathbb{N}(\mathcal{U}_1) = 0 \text{ and } \mathbb{N}(\mathcal{U}_2) = 0\} \\ &= \Pr\{\mathbb{N}(\mathcal{U}_1) = 0 \text{ and } \mathbb{N}(\mathcal{U}_2) = 0\} - \Pr\{\mathbb{N}(\mathcal{U}_1) = 0 \text{ and } \mathbb{N}(\mathcal{U}_2) = 0 \text{ and } \mathbb{N}(\mathcal{U}_3) = 0\} \\ &= \Pr\{\mathbb{N}(\mathcal{W} - (\mathcal{U}_1 \cup \mathcal{U}_2)) = k - 1\} - \Pr\{\mathbb{N}(\mathcal{W} - (\mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3)) = k - 1\} \end{aligned}$$

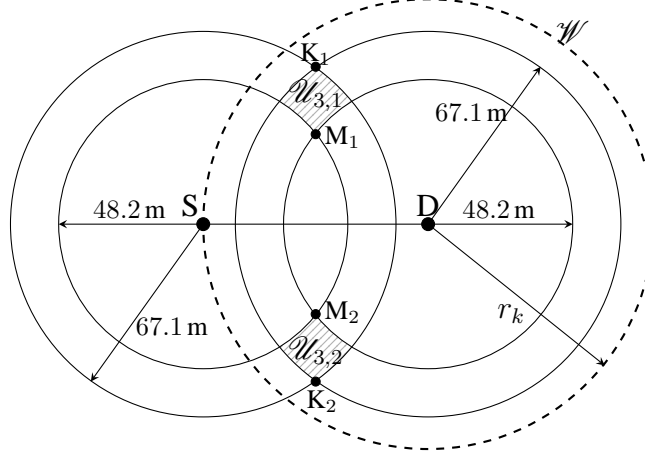


Fig. 4. A typical Type \mathcal{C} link making use of a Tier 3 helper, $67.1 \leq r_k \leq 74.7$.

$$\begin{aligned}
 &= \left(\frac{\mathcal{S}(\mathcal{W}) - (\mathcal{S}(\mathcal{U}_1) + \mathcal{S}(\mathcal{U}_2))}{\mathcal{S}(\mathcal{W})} \right)^{k-1} - \left(\frac{\mathcal{S}(\mathcal{W}) - (\mathcal{S}(\mathcal{U}_1) + \mathcal{S}(\mathcal{U}_2) + \mathcal{S}(\mathcal{U}_3))}{\mathcal{S}(\mathcal{W})} \right)^{k-1} \\
 &= \left(1 - \frac{\mathcal{S}(\mathcal{U}_1) + \mathcal{S}(\mathcal{U}_2)}{\pi r_k^2} \right)^{k-1} - \left(1 - \frac{\mathcal{S}(\mathcal{U}_1) + \mathcal{S}(\mathcal{U}_2) + \mathcal{S}(\mathcal{U}_3)}{\pi r_k^2} \right)^{k-1}
 \end{aligned} \tag{21}$$

where $\mathcal{S}(\mathcal{U}_3) = \mathbb{A}(67.1, 67.1, r_k) - 2\mathbb{A}(67.1, 48.2, r_k) + \mathbb{A}(48.2, 48.2, r_k)$, and $\mathbb{A}(r_1, r_2, \ell)$ was defined in (16a) and (16b). The maximum and minimum of $\mathcal{P}_{\text{Coop}}^{\text{Succ}, \mathcal{C}, 3}$ in this case are easy to obtain. Indeed, using eq. (4a) along with the fact that $\mathbb{Q}(x)$ is a monotonically decreasing function of x , we can readily see that

$$\mathbb{G}(67.1, 67.1) \leq \mathcal{P}_{\text{Coop}}^{\text{Succ}, \mathcal{C}, 3} \leq \mathbb{G}(48.2, 48.2). \tag{22}$$

Hence, the cooperative throughput of a Type \mathcal{C} link that utilizes a Tier 3 helper, i.e., $\mathcal{T}_{\text{Coop}}^{\mathcal{C}, 3}$, can be bounded as

$$\mathcal{L}_{\text{Coop}}^{\mathcal{C}, 3} \leq \mathcal{T}_{\text{Coop}}^{\mathcal{C}, 3} \leq \mathcal{U}_{\text{Coop}}^{\mathcal{C}, 3} \tag{23a}$$

$$\mathcal{L}_{\text{Coop}}^{\mathcal{C}, 3} \triangleq \mathbb{G}(67.1, 67.1) \times 2.75 \text{ (Mbps)} \tag{23b}$$

$$\mathcal{U}_{\text{Coop}}^{\mathcal{C}, 3} \triangleq \mathbb{G}(48.2, 48.2) \times 2.75 \text{ (Mbps)}. \tag{23c}$$

Using the results given in Subsections V-A through V-C, one can readily see

$$\mathcal{L}^{\mathcal{C}}(r_k) \leq \mathcal{T}^{\mathcal{C}}(r_k) \leq \mathcal{U}^{\mathcal{C}}(r_k) \tag{24a}$$

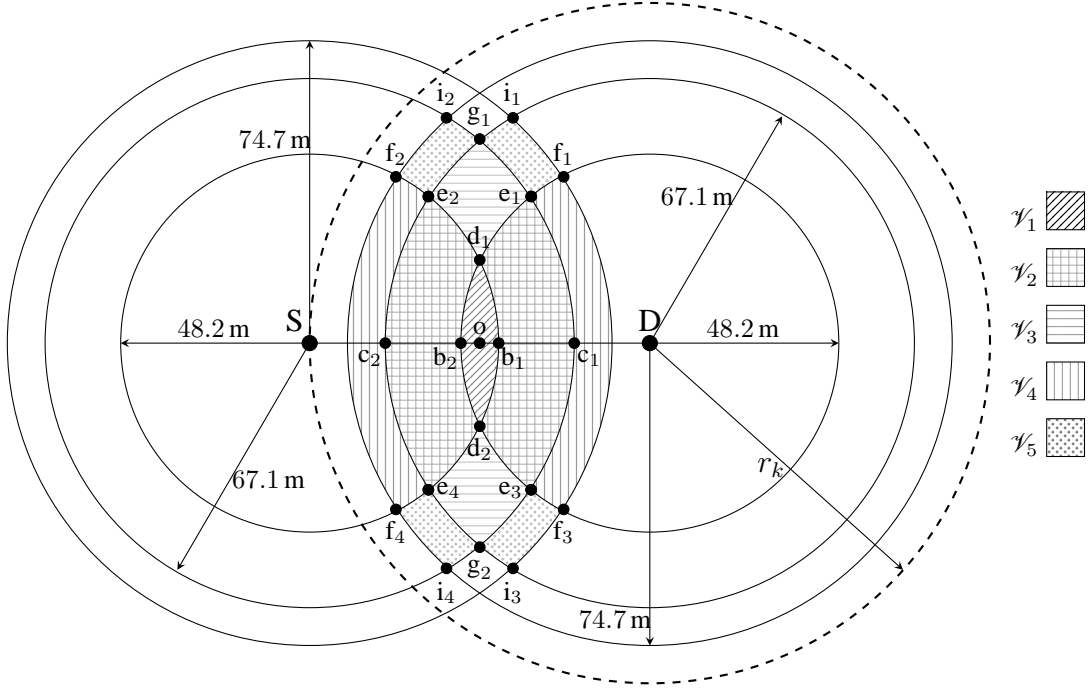


Fig. 5. A typical Type \mathcal{D} link with $74.7 < r_k \leq 96.4$ and the operating regions of Tiers 1 through 5 helpers.

where

$$\mathcal{L}^{\mathcal{C}}(r_k) = \sum_{i=1}^3 \mathcal{P}_{\mathcal{C},i} \mathcal{L}_{\text{Coop}}^{\mathcal{C},i} + \left(1 - \sum_{i=1}^3 \mathcal{P}_{\mathcal{C},i}\right) \mathcal{P}_{\text{Direct}}^{\text{Succ}}(r_k) \times 2 \text{ (Mbps)} \quad (24b)$$

$$\mathcal{U}^{\mathcal{C}}(r_k) = \sum_{i=1}^3 \mathcal{P}_{\mathcal{C},i} \mathcal{U}_{\text{Coop}}^{\mathcal{C},i} + \left(1 - \sum_{i=1}^3 \mathcal{P}_{\mathcal{C},i}\right) \mathcal{P}_{\text{Direct}}^{\text{Succ}}(r_k) \times 2 \text{ (Mbps)}. \quad (24c)$$

In consequence, the average throughput of a Type \mathcal{C} link can be bounded as

$$\overline{\mathcal{L}}^{\mathcal{C}} \leq \overline{\mathcal{T}}^{\mathcal{C}} \leq \overline{\mathcal{U}}^{\mathcal{C}} \quad (25a)$$

where

$$\overline{\mathcal{L}}^{\mathcal{C}} = \int_{67.1}^{74.7} \mathcal{L}^{\mathcal{C}}(r) f_{r_k}(r) dr \quad (25b)$$

$$\overline{\mathcal{U}}^{\mathcal{C}} = \int_{67.1}^{74.7} \mathcal{U}^{\mathcal{C}}(r) f_{r_k}(r) dr. \quad (25c)$$

VI. BOUNDS ON THE AVERAGE THROUGHPUT OF A TYPE \mathcal{D} LINK

In this section, we use the analysis given in Section V to obtain the maximum and minimum cooperative throughputs of a Type \mathcal{D} link. As mentioned in Subsection III-4, a Type \mathcal{D} link whose source and destination are more than 96.4 meters apart (i.e., $96.4 < r_k \leq 100$), can not use a Tier 1 helper. Therefore, we divide our analysis into two parts, viz., $74.7 < r_k \leq 96.4$ and $96.4 < r_k \leq 100$. Recall that in our helper selection algorithm, the lower the tier index, the higher the selection priority. Hence, using the procedure outlined in Section V for evaluating $\mathcal{P}_{\mathcal{C},1}$ through $\mathcal{P}_{\mathcal{C},3}$, we can obtain the probability that a Type \mathcal{D} link chooses a Tier i helper as

$$\mathcal{P}_{\mathcal{D},i} = \begin{cases} 1 - \left(1 - \frac{\mathcal{S}(\mathcal{V}_1)}{\mathcal{S}(\mathcal{W})}\right)^{k-1}, & i = 1 \\ \left(1 - \frac{\sum_{\ell=1}^{i-1} \mathcal{S}(\mathcal{V}_\ell)}{\mathcal{S}(\mathcal{W})}\right)^{k-1} - \left(1 - \frac{\sum_{\ell=1}^i \mathcal{S}(\mathcal{V}_\ell)}{\mathcal{S}(\mathcal{W})}\right)^{k-1}, & i = 2, \dots, 5 \end{cases} \quad (26)$$

where $\mathcal{S}(\mathcal{W}) = \pi r_k^2$ and $\mathcal{S}(\mathcal{V}_i)$ depends on r_k and should be evaluated for $74.7 < r_k \leq 96.4$ and $96.4 \leq r_k \leq 100$, separately. For the case where $74.7 < r_k \leq 96.4$, we can readily see from Fig. 5 and (16a) that

$$\mathcal{S}(\mathcal{V}_1) = \mathbb{A}(48.2, 48.2, r_k), \quad (27a)$$

$$\mathcal{S}(\mathcal{V}_2) = 2(\mathbb{A}(48.2, 67.1, r_k) - \mathcal{S}(\mathcal{V}_1)), \quad (27b)$$

$$\mathcal{S}(\mathcal{V}_3) = \mathbb{A}(67.1, 67.1, r_k) - \mathcal{S}(\mathcal{V}_2) - \mathcal{S}(\mathcal{V}_1), \quad (27c)$$

$$\mathcal{S}(\mathcal{V}_4) = 2(\mathbb{A}(48.2, 74.7, r_k) - \mathcal{S}(\mathcal{V}_1)) - \mathcal{S}(\mathcal{V}_2), \quad (27d)$$

$$\mathcal{S}(\mathcal{V}_5) = 2(\mathbb{A}(67.1, 74.7, r_k) - \mathbb{A}(67.1, 67.1, r_k)) - \mathcal{S}(\mathcal{V}_4). \quad (27e)$$

Recalling that there is no Tier 1 helper (or analogously $\mathcal{V}_1 = \emptyset$) for the case where $96.4 < r_k \leq 100$, and considering Fig. 6, we have

$$\mathcal{S}(\mathcal{V}_1) = 0. \quad (28)$$

In this case, $\mathcal{S}(\mathcal{V}_i)$, $i = 2, \dots, 5$ can be obtained from (27b) through (27e), respectively. In the next two subsections we obtain upper and lower bounds of the cooperative throughput for the cases where $74.7 < r_k \leq 96.4$ and $96.4 < r_k \leq 100$.

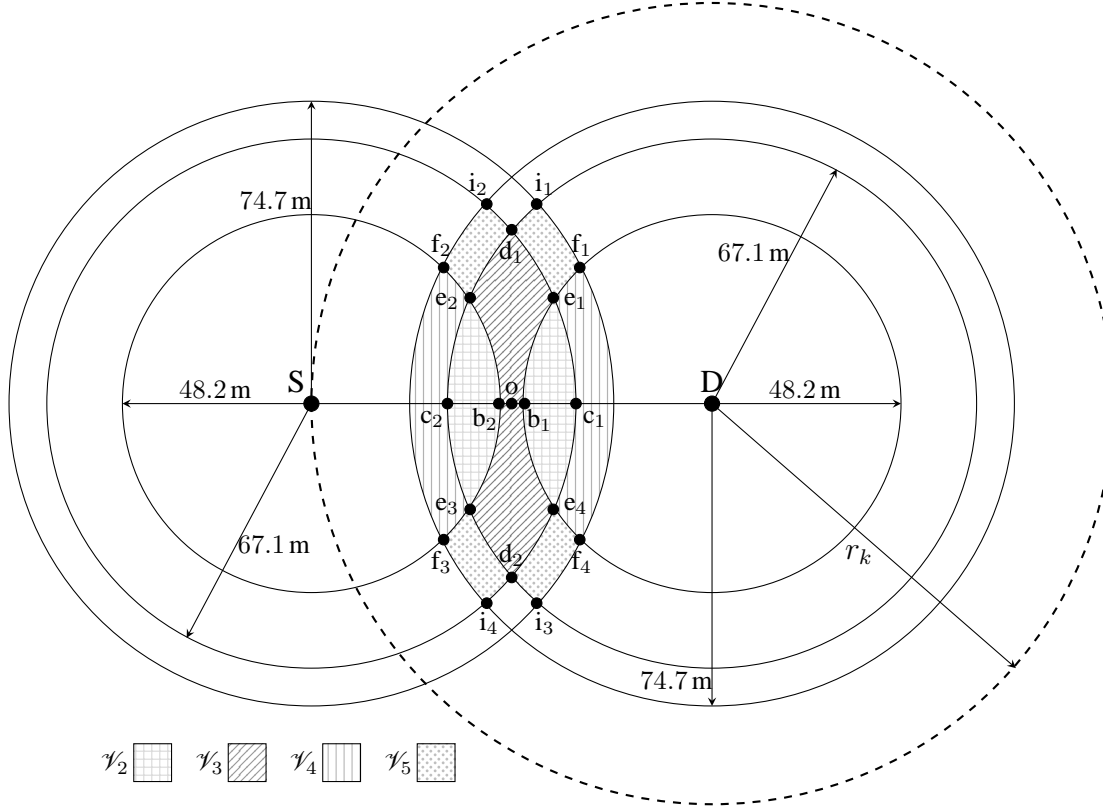


Fig. 6. A typical Type \mathcal{D} link with $96.4 < r_k \leq 100$ and the operating regions of Tiers 2 through 5 helpers.

A. Bounds on the Cooperative Throughput for $74.7 < r_k \leq 96.4$

When $74.7 < r_k \leq 96.4$, a helper from each of the tiers shown in Table III (Tiers 1 through 5) can be used to increase the transmission rate between S and D nodes. Comparing Fig. 5 with Figs. 2, 3 and 4, one can readily see that the operating regions of Tier 1, 2 and 3 helpers for a Type \mathcal{D} link are quite similar to those of Tier 1, 2 and 3 helpers for a Type \mathcal{C} link, respectively. Furthermore, the operating region of a Tier 4 helper for a Type \mathcal{D} link is similar to that of a Tier 2 helper for a Type \mathcal{C} link. Note that the operating region of a Tier 5 helper for a Type \mathcal{D} link is different from those of a Type \mathcal{C} link. However, using a procedure similar to that presented in Subsection V-C for a Tier 3 helper, we can readily evaluate the maximum and minimum of the cooperative throughput in this case. As a result, we can write

$$\mathcal{L}_{\text{Coop}}^{\mathcal{D},i} \leq \mathcal{T}_{\text{Coop}}^{\mathcal{D},i} \leq \mathcal{U}_{\text{Coop}}^{\mathcal{D},i}, \quad i = 1, \dots, 5 \quad (29)$$

TABLE IV
ILLUSTRATION OF THE POSITIONS OF THE HELPERS ACHIEVING MAXIMUM AND MINIMUM $\mathcal{P}_{\text{Coop}}^{\text{Succ}}$ AND THEIR
CORRESPONDING THROUGHPUTS FOR A TYPE \mathcal{D} LINK WITH $74.7 < r_k \leq 96.4$.

Helper's Tier	Point(s) with max. $\mathcal{P}_{\text{Coop}}^{\text{Succ}}$	Points with min. $\mathcal{P}_{\text{Coop}}^{\text{Succ}}$	$\mathcal{L}_{\text{Coop}}^{\mathcal{D}_1, i}$ (Mbps)	$\mathcal{U}_{\text{Coop}}^{\mathcal{D}_1, i}$ (Mbps)
1	o	d ₁ , d ₂	$\mathbb{G}(48.2, 48.2) \times 5.5$	$\mathbb{G}\left(\frac{r_k}{2}, \frac{r_k}{2}\right) \times 5.5$
2	b ₁ , b ₂	e ₁ , e ₂ , e ₃ , e ₄	$\mathbb{G}(48.2, 67.1) \times 3.67$	$\mathbb{G}(48.2, r_k - 48.2) \times 3.67$
3	c ₁ , c ₂	g ₁ , g ₂	$\mathbb{G}(67.1, 67.1) \times 2.75$	$\mathbb{G}(48.2, 48.2) \times 2.75$
4	d ₁ , d ₂	f ₁ , f ₂ , f ₃ , f ₄	$\mathbb{G}(48.2, 74.7) \times 1.69$	$\mathbb{G}(67.1, r_k - 67.1) \times 1.69$
5	e ₁ , e ₂ , e ₃ , e ₄	i ₁ , i ₂ , i ₃ , i ₄	$\mathbb{G}(67.1, 74.7) \times 1.47$	$\mathbb{G}(48.2, 67.1) \times 1.47$

TABLE V
ILLUSTRATION OF THE POSITIONS OF THE HELPERS ACHIEVING MAXIMUM AND MINIMUM $\mathcal{P}_{\text{Coop}}^{\text{Succ}}$ AND THEIR
CORRESPONDING THROUGHPUTS FOR A TYPE \mathcal{D} LINK WITH $96.4 < r_k \leq 100$.

Helper's Tier	Point(s) with max. $\mathcal{P}_{\text{Coop}}^{\text{Succ}}$	Points with min. $\mathcal{P}_{\text{Coop}}^{\text{Succ}}$	$\mathcal{L}_{\text{Coop}}^{\mathcal{D}_2, i}$ (Mbps)	$\mathcal{U}_{\text{Coop}}^{\mathcal{D}_2, i}$ (Mbps)
2	b ₁ , b ₂	e ₁ , e ₂ , e ₃ , e ₄	$\mathbb{G}(48.2, 67.1) \times 3.67$	$\mathbb{G}(48.2, r_k - 48.2) \times 3.67$
3	o	d ₁ , d ₂	$\mathbb{G}(67.1, 67.1) \times 2.75$	$\mathbb{G}\left(\frac{r_k}{2}, \frac{r_k}{2}\right) \times 2.75$
4	c ₁ , c ₂	f ₁ , f ₂ , f ₃ , f ₄	$\mathbb{G}(48.2, 74.7) \times 1.69$	$\mathbb{G}(67.1, r_k - 67.1) \times 1.69$
5	e ₁ , e ₂ , e ₃ , e ₄	i ₁ , i ₂ , i ₃ , i ₄	$\mathbb{G}(67.1, 74.7) \times 1.47$	$\mathbb{G}(48.2, 67.1) \times 1.47$

where $\mathcal{L}_{\text{Coop}}^{\mathcal{D}_1, i}$ and $\mathcal{U}_{\text{Coop}}^{\mathcal{D}_1, i}$ ($i = 1, \dots, 5$) are shown in Table IV and \mathcal{D}_1 denotes a Type \mathcal{D} link with $74.7 < r_k \leq 96.4$. Note also that Table IV shows the locations of helpers in Fig. 5 that can achieve the maximum and minimum $\mathcal{P}_{\text{Coop}}^{\text{Succ}}$ for each tier. As a result, the cooperative throughput of a Type \mathcal{D} link for $74.7 < r_k \leq 96.4$ can be bounded as

$$\mathcal{L}^{\mathcal{D}_1}(r_k) \leq \mathcal{T}^{\mathcal{D}_1}(r_k) \leq \mathcal{U}^{\mathcal{D}_1}(r_k) \quad (30a)$$

where

$$\mathcal{L}^{\mathcal{D}_1}(r_k) = \sum_{i=1}^5 \mathcal{P}_{\mathcal{D}, i} \mathcal{L}_{\text{Coop}}^{\mathcal{D}_1, i} + \left(1 - \sum_{i=1}^5 \mathcal{P}_{\mathcal{D}, i}\right) \mathcal{P}_{\text{Direct}}^{\text{Succ}}(r_k) \times 1 \text{ (Mbps)} \quad (30b)$$

$$\mathcal{U}^{\mathcal{D}_1}(r_k) = \sum_{i=1}^5 \mathcal{P}_{\mathcal{D}, i} \mathcal{U}_{\text{Coop}}^{\mathcal{D}_1, i} + \left(1 - \sum_{i=1}^5 \mathcal{P}_{\mathcal{D}, i}\right) \mathcal{P}_{\text{Direct}}^{\text{Succ}}(r_k) \times 1 \text{ (Mbps)}. \quad (30c)$$

Averaging over the distribution of r_k we obtain

TABLE VI
THE SIMULATION PARAMETERS.

Parameter	P_t	P_{th}	α	σ_ψ	K	RTS	CoopRTS	CTS	HTS	data
Value	1 mW	-98 dBm	3	6 dB	-40 dB	352 bits	352 bits	304 bits	304 bits	1000 bytes

$$\overline{\mathcal{L}}^{\mathcal{D}_1} \leq \overline{\mathcal{T}}^{\mathcal{D}_1} \leq \overline{\mathcal{U}}^{\mathcal{D}_1} \quad (31a)$$

where

$$\overline{\mathcal{L}}^{\mathcal{D}_1} = \int_{74.7}^{96.4} \mathcal{L}^{\mathcal{D}_1}(r) f_{r_k}(r) dr \quad (31b)$$

$$\overline{\mathcal{U}}^{\mathcal{D}_1} = \int_{74.7}^{96.4} \mathcal{U}^{\mathcal{D}_1}(r) f_{r_k}(r) dr \quad (31c)$$

and $\overline{\mathcal{T}}^{\mathcal{D}_1}$ is the average throughput of a Type \mathcal{D} link for the case where $74.7 < r_k \leq 96.4$.

B. Bounds on the Cooperative Throughput for $96.4 < r_k \leq 100$

The analysis in this subsection is similar to what presented in Subsection VI-A except that in this case a Tier 1 helper cannot be used as mentioned in Subsection III-4. Therefore, one has

$$\mathcal{L}_{\text{Coop}}^{\mathcal{D}_{2,i}} \leq \mathcal{T}_{\text{Coop}}^{\mathcal{D}_{2,i}} \leq \mathcal{U}_{\text{Coop}}^{\mathcal{D}_{2,i}}, \quad i = 2, \dots, 5 \quad (32)$$

where $\mathcal{L}_{\text{Coop}}^{\mathcal{D}_{2,i}}$ and $\mathcal{U}_{\text{Coop}}^{\mathcal{D}_{2,i}}$, $i = 2, \dots, 5$ are shown in Table V for each helper's tier. Also shown in this table are the points at which the maximum and minimum $\mathcal{P}_{\text{Coop}}^{\text{Succ}}$ are achieved for each tier. Hence, we can write

$$\mathcal{L}^{\mathcal{D}_2}(r_k) \leq \mathcal{T}^{\mathcal{D}_2}(r_k) \leq \mathcal{U}^{\mathcal{D}_2}(r_k) \quad (33a)$$

where

$$\mathcal{L}^{\mathcal{D}_2}(r_k) = \sum_{i=2}^5 \mathcal{P}_{\mathcal{D},i} \mathcal{L}_{\text{Coop}}^{\mathcal{D}_{2,i}} + \left(1 - \sum_{i=2}^5 \mathcal{P}_{\mathcal{D},i}\right) \mathcal{P}_{\text{Direct}}^{\text{Succ}}(r_k) \times 1 \text{ (Mbps)} \quad (33b)$$

$$\mathcal{U}^{\mathcal{D}_2}(r_k) = \sum_{i=2}^5 \mathcal{P}_{\mathcal{D},i} \mathcal{U}_{\text{Coop}}^{\mathcal{D}_{2,i}} + \left(1 - \sum_{i=2}^5 \mathcal{P}_{\mathcal{D},i}\right) \mathcal{P}_{\text{Direct}}^{\text{Succ}}(r_k) \times 1 \text{ (Mbps)}. \quad (33c)$$

Using (33), we can find the upper and lower bounds of the average throughput of a Type \mathcal{D} link for the case where $96.4 < r_k \leq 100$ as

$$\overline{\mathcal{L}}^{\mathcal{D}_2} \leq \overline{\mathcal{T}}^{\mathcal{D}_2} \leq \overline{\mathcal{U}}^{\mathcal{D}_2} \quad (34a)$$

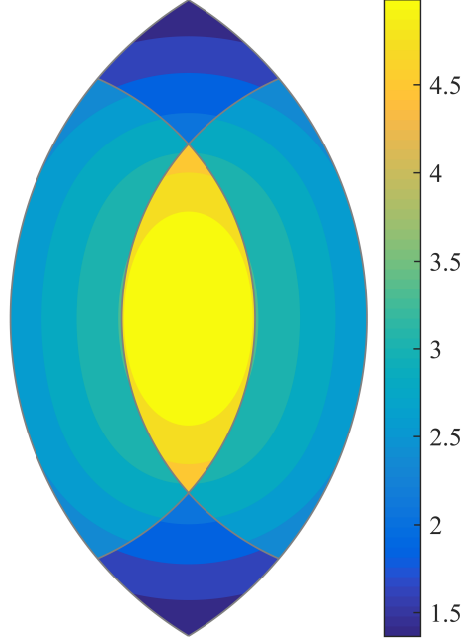


Fig. 7. The contour plot of the cooperative throughput of a Type C link achieved through Tier 1, 2 and 3 helpers for $67.1 < r_k \leq 74.7$. The throughputs are in Mbps.

where

$$\overline{\mathcal{L}}^{\mathcal{D}_2} = \int_{96.4}^{100} \mathcal{L}^{\mathcal{D}_2}(r) f_{r_k}(r) dr \quad (34b)$$

$$\overline{\mathcal{U}}^{\mathcal{D}_2} = \int_{96.4}^{100} \mathcal{U}^{\mathcal{D}_2}(r) f_{r_k}(r) dr. \quad (34c)$$

In summary, we can combine eqs. (10), (11), (25), (31) and (34) to bound the average throughput of our proposed CoopMAC protocol in the presence of shadowing as

$$\overline{\mathcal{L}} \leq \overline{\mathcal{T}} \leq \overline{\mathcal{U}} \quad (35a)$$

where

$$\overline{\mathcal{L}} = \overline{\mathcal{L}}^{\mathcal{D}_2} + \overline{\mathcal{L}}^{\mathcal{D}_1} + \overline{\mathcal{L}}^{\mathcal{C}} + \mathcal{T}^{\mathcal{A}} + \mathcal{T}^{\mathcal{B}} \quad (35b)$$

$$\overline{\mathcal{U}} = \overline{\mathcal{U}}^{\mathcal{D}_2} + \overline{\mathcal{U}}^{\mathcal{D}_1} + \overline{\mathcal{U}}^{\mathcal{C}} + \mathcal{T}^{\mathcal{A}} + \mathcal{T}^{\mathcal{B}}. \quad (35c)$$

VII. NUMERICAL RESULTS

We have used computer simulation to evaluate the throughput performance of our proposed CoopMAC scheme and illustrate its superiority over the conventional CoopMAC protocol in which the helpers are selected randomly (i.e., the locations of helpers are not taken into account). As mentioned earlier, only Type \mathcal{C} and Type \mathcal{D} links can take advantage of cooperation and, therefore, we have only considered these link types in our analysis. Table VI shows the parameters that have been used in our computer simulations. Throughout this section, we assume these parameters remain unchanged unless otherwise specified. The simulation results have been obtained using the Monte-Carlo method for two million independent realizations of a network whose nodes are distributed according to a two-dimensional PPP with density $0.0005 \leq \lambda \leq 0.005$.

Fig. 7 shows a contour plot of the cooperative throughput achieved by a Type \mathcal{C} link. Clearly, the achievable throughput for most of Tier 1 helpers is greater than 4.5 Mbps (and, actually, very close to $\mathbb{G}\left(\frac{r_k}{2}, \frac{r_k}{2}\right) \times 5.5$ Mbps), and for a small fraction of these helpers the throughput is smaller than 4.5 Mbps. For Tier 2 helpers, the maximum achievable throughput is approximately 1 Mbps less than the minimum throughput that can be achieved by a Tier 1 helper. This explains why a Tier 1 helper is superior to a Tier 2 helper in our proposed scheme. Note that the throughput achieved by Tier 3 helpers is less than 50% of that of the Tier 1 helpers. For this reason, we give them the lowest priority in our proposed scheme.

The average throughput of a Type \mathcal{C} link making use of the proposed and the conventional CoopMAC schemes as a function of λ are shown in Fig. 8. Also shown in this figure are the upper and lower bounds derived in (25) for a Type \mathcal{C} link. As seen in this figure, the throughput improvement due to proposed scheme is quite significant. Observe that when λ increases, the throughput of the proposed scheme becomes very close to the upper bound. This is because when λ increases the chance of finding a helper near the best helper (i.e., a helper located halfway between source and destination) also increases. Fig. 8 also shows that for a Type \mathcal{C} link utilizing the conventional CoopMAC protocol the average throughput is slightly larger than the lower bound. To explain this, we note from Fig. 7 that a randomly selected helper is more likely to be from Tiers 2 and 3 and the cooperative throughput that can be achieved through a helper from these tiers is generally closer to the lower bound than the upper bound.

The cooperative throughputs of a Type \mathcal{D} link are illustrated as contour maps in Fig. 9a

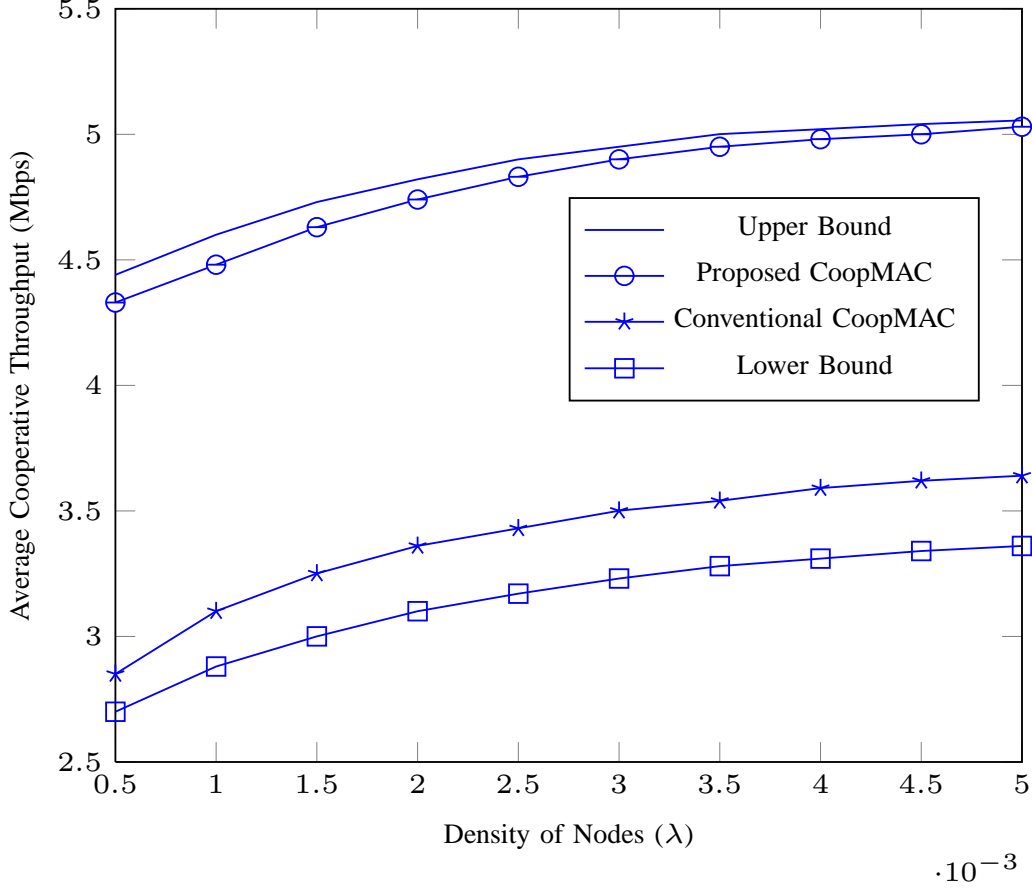


Fig. 8. The average throughput as a function of λ for a Type C link making use of the proposed and conventional CoopMAC protocols.

for $74.7 < r_k \leq 96.4$ and in Fig. 9b and for $96.4 < r_k \leq 100$. Clearly, for the case where $74.7 < r_k \leq 96.4$, the achievable cooperative throughput can be as large as 4.6 Mbps whereas for $96.4 < r_k \leq 100$ the maximum cooperative throughput is approximately 2.8 Mbps. As mentioned in Section VI-A, this difference has its roots in the fact that in the former case a Tier 1 helper can be taken advantage of whereas in the latter it cannot. Note, importantly, that in Fig. 9a the maximum cooperative throughput can be achieved through the helpers that are no farther than 48.2 meters from the source and destination nodes. Since only a small number of helpers have this property, a selection scheme that does not account for the locations of helpers is unlikely to select one of these helpers and, thus, achieve the maximum throughput.

The average cooperative throughput of a Type D link making use of the proposed and conventional CoopMAC protocol is shown in Fig. 10. Both $74.7 < r_k \leq 96.4$ and $96.4 < r_k \leq 100$

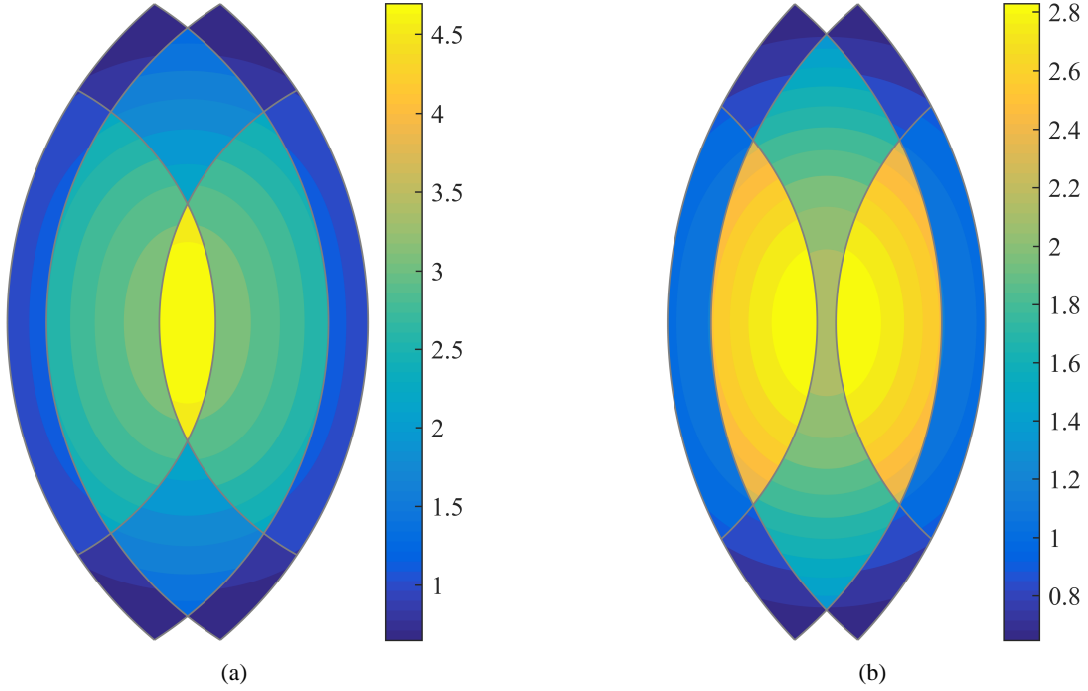


Fig. 9. The contour plots of the cooperative throughput of a Type \mathcal{D} link for (a) $74.7 < r_k \leq 96.4$, and (b) $96.4 < r_k \leq 100$. The throughputs are in Mbps.

cases are considered. Similar to Fig. 8, in this figure the average throughput of the proposed scheme is close to the upper bound particularly when λ approaches 0.005. Note that the average throughput of conventional CoopMAC scheme is slightly larger than the lower bound. This is due to the fact that most of the cooperative throughputs illustrated in Figs. 9a and 9b are closer to the lower bound than the upper bound. Hence, random selection of the helpers results in an average throughput that is close to the lower bound.

Fig. 11 shows the average throughput as a function of λ achieved by the proposed and conventional CoopMAC schemes for all link types. Observe that in this case, Type \mathcal{A} and \mathcal{B} links also contribute to the average throughput. Therefore, the average throughput is approximately twice as large as it was for Type \mathcal{C} and \mathcal{D} links. Similar to Figs. 8 and 10, by increasing the density of nodes, the average throughput of the proposed scheme becomes closer to the upper bound in contrast to the conventional CoopMAC whose throughput performance does not improve much.

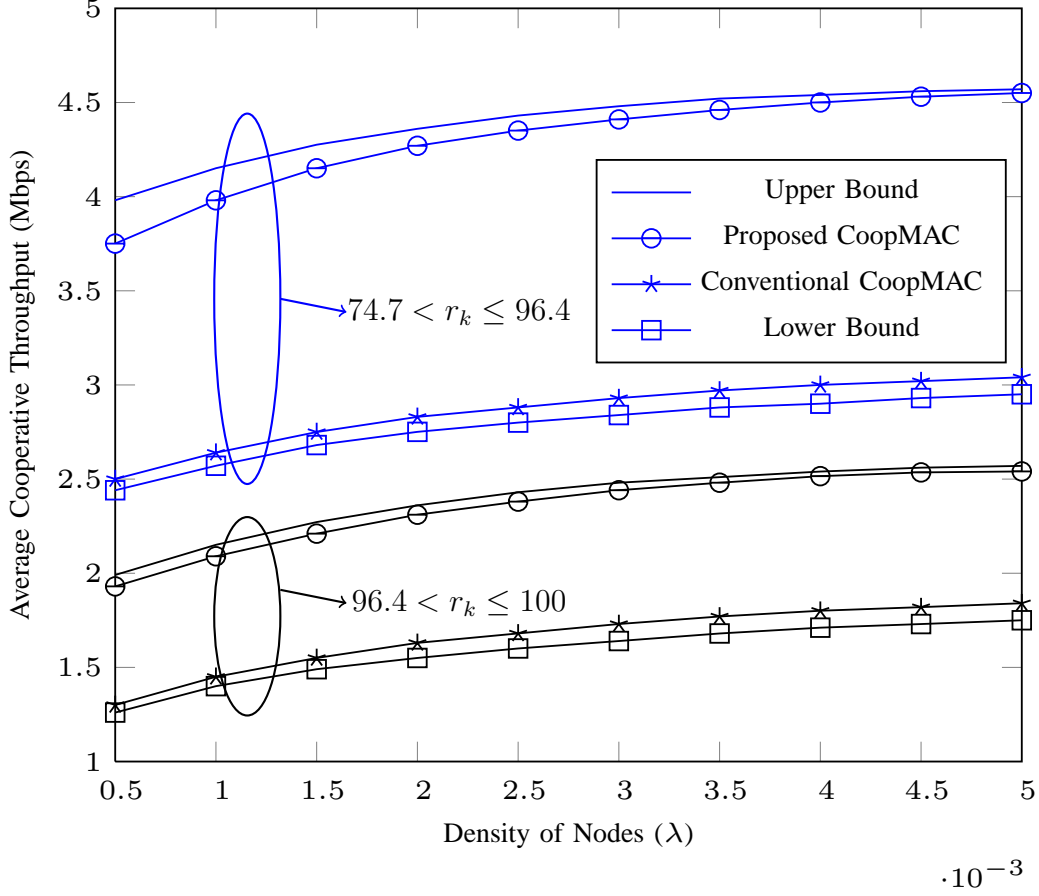


Fig. 10. The average cooperative throughput as a function of λ for a Type \mathcal{D} link making use of the proposed and conventional CoopMAC protocols.

VIII. CONCLUSION

In this paper, we considered a CoopMAC network based on the IEEE 802.11b Standard, and studied its throughput performance in the presence of shadowing and spatially distributed random nodes. We first identified four link types according to their achievable throughput and divided the potential helpers for each link type into several tiers based on the cooperative throughput that they can provide. Then, the locus of the helpers in each tier were determined using simple algebraic expressions. In our proposed CoopMAC protocol, the helpers with the lowest tier index have the highest priority to be selected for cooperation. We derived upper and lower bounds on the average throughput of different link types in the network. Our numerical results illustrated the superiority of our scheme over the conventional CoopMAC protocol. Indeed, in all the examined scenarios the average throughput of the proposed scheme was very close to the upper bound

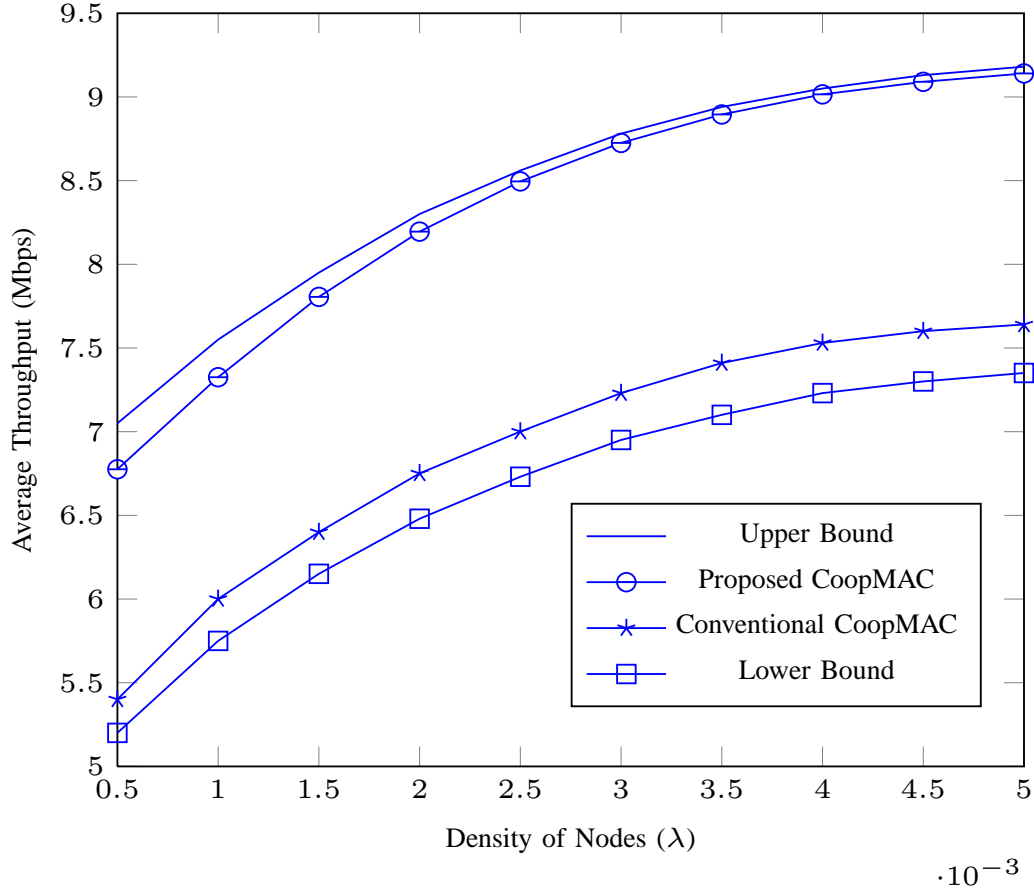


Fig. 11. The average throughputs of all link types as a function of λ for the proposed and conventional CoopMAC schemes.

while the average throughput of the conventional scheme was slightly larger than the lower bound.

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APPENDIX A

PROOF OF LEMMA 1

Recalling from Table II that the transmission rates of all Tier 1 helpers for a Type \mathcal{C} link are equal to 5.5 Mbps, and that $\mathcal{T}_{\text{Coop}}^{\mathcal{C},1} = R_{\text{Coop}} \mathcal{P}_{\text{Coop}}^{\text{Succ},\mathcal{C},1}$, we only need to find the maximum and minimum of $\mathcal{P}_{\text{Coop}}^{\text{Succ},\mathcal{C},1}$. To this end, we first note from Fig. 2 that a helper is in \mathcal{U}_1 if

$$d_{\text{SH}} \leq 48.2 \tag{36a}$$

$$d_{\text{HD}} \leq 48.2 \quad (36\text{b})$$

$$r_k \leq d_{\text{SH}} + d_{\text{HD}}. \quad (36\text{c})$$

In order to maximize $\mathcal{P}_{\text{Coop}}^{\text{Succ}, \mathcal{C}, 1} = \mathbb{G}(d_{\text{SH}}, d_{\text{HD}})$, one has to minimize the arguments of both \mathbb{Q} -functions in (4a) owing to the fact that the Gaussian \mathbb{Q} -function is strictly decreasing in its argument. Hence, one has to minimize d_{SH} and d_{HD} provided that the constraints (36a) through (36c) are satisfied. We first use proof by contradiction to show that the helper which maximizes $\mathcal{P}_{\text{Coop}}^{\text{Succ}, \mathcal{C}, 1}$ should be located on SD line segment in Fig. 2. We then prove that the best helper is indeed located halfway between S and D.

Suppose that H^* is a helper in \mathcal{U}_1 that maximizes $\mathcal{P}_{\text{Coop}}^{\text{Succ}, \mathcal{C}, 1}$ and is not located on SD line segment, i.e., $r_k < d_{\text{SH}^*} + d_{\text{H}^*\text{D}}$. Assume now that H^\perp is the projection of H^* on SD line segment and, thus, $r_k = d_{\text{SH}^\perp} + d_{\text{H}^\perp\text{D}}$. It is clear that, $d_{\text{SH}^\perp} < d_{\text{SH}^*} \leq 48.2$ and $d_{\text{H}^\perp\text{D}} < d_{\text{H}^*\text{D}} \leq 48.2$. Recalling that $\mathbb{Q}(x)$ is a monotonically decreasing function of x , we see from (4a) that $\mathbb{G}(d_{\text{SH}^*}, d_{\text{H}^*\text{D}}) < \mathbb{G}(d_{\text{SH}^\perp}, d_{\text{H}^\perp\text{D}})$. Hence, our initial assumption that H^* maximizes $\mathcal{P}_{\text{Coop}}^{\text{Succ}, \mathcal{C}, 1}$ is wrong and the helper with maximum $\mathcal{P}_{\text{Coop}}^{\text{Succ}, \mathcal{C}, 1}$ has to be located on SD line segment, i.e., (36c) should be changed to $d_{\text{SH}} + d_{\text{HD}} = r_k$ for this helper. Substituting for d_{HD} by $r_k - d_{\text{SH}}$ in (4a) we obtain

$$\mathbb{G}(d_{\text{SH}}, r_k - d_{\text{SH}}) = \mathbb{Q}(\nu + \mu \log_{10}(d_{\text{SH}})) \times \mathbb{Q}(\nu + \mu \log_{10}(r_k - d_{\text{SH}})). \quad (37)$$

Observe that for $0 < d_{\text{SH}} < r_k$, $\mathbb{Q}(\nu + \mu \log_{10}(d_{\text{SH}}))$ is a decreasing function of d_{SH} whereas $\mathbb{Q}(\nu + \mu \log_{10}(r_k - d_{\text{SH}}))$ is an increasing function of d_{SH} . In addition, the arguments of both \mathbb{Q} -functions are negative and, thus, they are both concave functions of d_{SH} . Consequently, the product of the \mathbb{Q} -functions in (37) is a concave function of d_{SH} provided that $0 < d_{\text{SH}} < r_k$ [20, Exercise 3.32 (b)]. Differentiating the right of (37) with respect to d_{SH} and equating it to zero we obtain $d_{\text{SH}} = \frac{r_k}{2}$. Thus, the maximum cooperative throughput in this case is obtained when the helper is located halfway between S and D (point M in Fig 2). Note that this result is optimum because it maximizes $\mathcal{P}_{\text{Coop}}^{\text{Succ}, \mathcal{C}, 1}$ and satisfies (36a) through (36c).

To obtain the minimum value of $\mathcal{P}_{\text{Coop}}^{\text{Succ}, \mathcal{C}, 1}$, one should maximize the arguments of both \mathbb{Q} -functions in (4a) so that the inequality constraints given in (36a) to (36c) are satisfied. Considering the fact that $67.1 \leq r_k \leq 74.7$, this occurs when $d_{\text{SH}} = d_{\text{HD}} = 48.2$ m, i.e., the helper is located

on K_1 or K_2 in Fig. 2. In summary, we can write

$$\mathbb{G}(48.2, 48.2) \leq \mathcal{P}_{\text{Coop}}^{\text{Succ}, \mathcal{C}, 1} \leq \mathbb{G}\left(\frac{r_k}{2}, \frac{r_k}{2}\right) \quad (38)$$

which results in (17).

APPENDIX B

PROOF OF LEMMA 2

As shown in Fig. 3, a Tier 2 helper should be located either in $\mathcal{U}_{2,1}$ which is characterized as

$$d_{\text{SH}} \leq 48.2 \quad (39a)$$

$$48.2 \leq d_{\text{HD}} \leq 67.1 \quad (39b)$$

$$r_k \leq d_{\text{SH}} + d_{\text{HD}} \quad (39c)$$

or in $\mathcal{U}_{2,2}$ characterized as

$$d_{\text{HD}} \leq 48.2 \quad (40a)$$

$$48.2 \leq d_{\text{SH}} \leq 67.1 \quad (40b)$$

$$r_k \leq d_{\text{SH}} + d_{\text{HD}}. \quad (40c)$$

Similar to the proof given for a Tier 1 helper, we first show that the maximum $\mathcal{P}_{\text{Coop}}^{\text{Succ}, \mathcal{C}, 2}$ is achieved through a helper that is located on the SD line segment. Again, we use a proof by contradiction. Assume that H^* is a helper in $\mathcal{U}_{2,1}$ through which the maximum $\mathcal{P}_{\text{Coop}}^{\text{Succ}, \mathcal{C}, 2}$ is achieved. Also assume that H' is another helper in $\mathcal{U}_{2,1}$ which is located on the SD line segment such that $d_{H^*D} = d_{H'D}$ and $\mathbb{G}(d_{\text{SH}^*}, d_{H^*D}) > \mathbb{G}(d_{\text{SH}'}, d_{H'D})$. It is clear that $r_k < d_{\text{SH}^*} + d_{H^*D}$ and $r_k = d_{\text{SH}'} + d_{H'D}$. Recalling that $d_{H^*D} = d_{H'D}$, one can readily see that $d_{\text{SH}'} < d_{\text{SH}^*}$ or analogously

$$\mathbb{Q}(\nu + \mu \log_{10}(d_{\text{SH}^*})) < \mathbb{Q}(\nu + \mu \log_{10}(d_{\text{SH}'})) \quad (41)$$

which follows from the fact that the Gaussian \mathbb{Q} -function is monotonically decreasing in its argument. In consequence, using (4b) we can obtain $\mathbb{G}(d_{\text{SH}^*}, d_{H^*D}) < \mathbb{G}(d_{\text{SH}'}, d_{H'D})$ which contradicts our initial assumption that $\mathbb{G}(d_{\text{SH}^*}, d_{H^*D}) > \mathbb{G}(d_{\text{SH}'}, d_{H'D})$. This argument is also

true for the case where the helpers are located in $\mathcal{U}_{2,2}$. As a result, the helper with maximum $\mathcal{P}_{\text{Coop}}^{\text{Succ},\mathcal{C},2}$ must be located on the SD line segment. Hence, the maximum $\mathcal{P}_{\text{Coop}}^{\text{Succ},\mathcal{C},2}$ for the best helper should be obtained from (37). As seen in Appendix A, the product of the \mathbb{Q} -functions on the right of (37) is a concave function of d_{SH} whose maximum is attained at $d_{\text{SH}} = \frac{r_k}{2}$. For a Tier 2 helper that is located on the SD line segment, d_{SH} cannot be equal to $\frac{r_k}{2}$. However, considering the concavity of $\mathbb{G}(d_{\text{SH}}, r_k - d_{\text{SH}})$ in d_{SH} for $0 < d_{\text{SH}} < r_k$, we conclude that the maximum of $\mathbb{G}(d_{\text{SH}}, r_k - d_{\text{SH}})$ is attained at $d_{\text{SH}} = r_k - 48.2$ (M_1 in Fig. 4), or at $d_{\text{SH}} = 48.2$ (M_2 in Fig. 4).

To obtain the minimum of $\mathcal{P}_{\text{Coop}}^{\text{Succ},\mathcal{C},2}$ we note from (4a) that the \mathbb{Q} -functions are both minimum when their arguments are maximum. This minimum is attained at $d_{\text{SH}} = 48.2$ and $d_{\text{HD}} = 67.1$ when the helper is located in $\mathcal{U}_{2,1}$ (K_1 and K_2 in Fig. 4), or at $d_{\text{SH}} = 67.1$ and $d_{\text{HD}} = 48.2$ when the helper is located in $\mathcal{U}_{2,2}$ (K_3 and K_4 in Fig. 4). Thus, $\mathcal{P}_{\text{Coop}}^{\text{Succ},\mathcal{C},2}$ can be bounded as

$$\mathbb{G}(48.2, 67.1) \leq \mathcal{P}_{\text{Coop}}^{\text{Succ},\mathcal{C},2} \leq \mathbb{G}(48.2, r_k - 48.2) \quad (42)$$

which leads to (19).